

PUZZLE SUITE · GAME THEORY NOTES

The Game Theory Behind Card Solver

Chance, hidden information and expected value across six card games

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A short game-theory primer

Game theory is the mathematics of decisions whose outcome depends on choices, chance, or other people. The ideas below come up throughout this page.

Perfect vs imperfect information

In a game of **perfect information**, every player sees everything that has happened so far, as in chess or Connect 4. In a game of **imperfect information**, something is hidden, such as an opponent's cards in poker. [1]

Zero-sum

A game is **zero-sum** when one player's gain is exactly the other's loss. Most two-player board games and heads-up poker are zero-sum. [2]

Minimax

Minimax means choosing the move whose *worst-case* result is best, on the assumption that the opponent always replies as well as possible. Von Neumann's 1928 minimax theorem showed that every finite two-player zero-sum game has a well-defined value when players are allowed to randomise. [3][4]

Nash equilibrium

A **Nash equilibrium** is a set of strategies in which no player can do better by changing only their own strategy. John Nash proved in 1950 that every finite game has at least one, possibly in mixed (randomised) strategies. In poker this is what "GTO" (game-theory optimal) refers to. [5][6]

Expected value (EV)

The **expected value** is the probability-weighted average outcome of a decision: what it would earn per play over many repetitions. When there is no opponent to outwit, the best play is usually just the play with the highest EV. [7]

Solved games

Following Allis, a game is **ultra-weakly solved** when its result under perfect play is known, **weakly solved** when a strategy that achieves that result from the start is known, and **strongly solved** when perfect play is known from every legal position. [8][9]

Poker (Texas hold'em)

IMPERFECT INFORMATION · ZERO-SUM

1 The game, in plain words

In Texas hold'em, each player receives two private **hole cards**. Five shared **community cards** are then dealt face up in three stages: three at once (the flop), then one (the turn), then one more (the river). There is a round of betting before the flop and after each stage. In each round, a player may **fold** (give up the hand), **call** (match the

current bet) or **raise** (increase it). If more than one player is left at the end, the best five-card hand made from hole and community cards wins the **pot**, the money bet so far.[10]

2 The game theory behind it

Poker is a game of **imperfect information**, because you cannot see your opponents' cards, and it also involves **chance**. You can't just compute "the best move" as in chess. You have to reason about what the opponent might hold, and about what your own actions reveal.

- **Equity** is the share of the pot you would win on average if all remaining cards were dealt with no more betting.
- **Pot odds** compare the price of a call with the size of the pot you would then be playing for. If calling costs C and the pot after your call would be P, calling has positive expected value when your equity is greater than C ÷ P. *Example:* if the pot is 100 and your opponent bets 50, you pay 50 to play for 200, so you need more than 25% equity to call.[11]
- A **range** is the full set of hands an opponent could plausibly hold, given how they have played. Strong players think about ranges, not single hands.
- A **Nash equilibrium** strategy, known in poker as **GTO** (game-theory optimal), cannot be exploited. Against it, no opposing strategy does better than the game's value. It is not necessarily the most profitable strategy against a weak player. It is the one that loses nothing to a perfect one.[5]

In 2015, Michael Bowling and colleagues at the University of Alberta reported in *Science* that **heads-up limit hold'em** (two players, fixed bet sizes) is **essentially weakly solved**. Their program, Cepheus, plays so close to a Nash equilibrium that the remaining gap could not be reliably detected even over a human lifetime of play. They computed it with a variant of **counterfactual regret minimisation** called CFR+, an algorithm that repeatedly improves both players' strategies by tracking, at each decision, how much better each alternative action would have done (its "regret").[12] No-limit games and games with more players are not solved.

3 Finding the best play, and how the app does it

Computing GTO strategies for full-size poker takes specialised solvers and serious computing power. For a single decision, the main inputs are equity against a range, the pot odds, and an estimate of how often the opponent will fold.

IN THE APP

Card Solver's fold, call or raise suggestion comes from a **simplified, clearly labelled model** built on hand strength and pot odds. It is **not a GTO solver** and does not compute a Nash equilibrium. Treat it as a teaching aid for thinking about EV, not as equilibrium play.

Blackjack

PLAYER VS FIXED DEALER RULES

1 The game, in plain words

In blackjack, you play against the dealer, not against other players. Cards 2–10 count at face value, jacks, queens and kings count 10, and an ace counts 1 or 11. You want a total closer to 21 than the dealer's without going over ("busting"). You can **hit** (take a card), **stand** (stop), **double down** (double your bet and take exactly

one more card), **split** a pair into two hands, and, where the rules allow, **surrender** (give up half your bet). A two-card 21 is a "blackjack" and usually pays 3:2. The dealer has no choices to make and must follow fixed drawing rules.[13]

2 The game theory behind it

Because the dealer's play is fixed, blackjack is **not a strategic contest**: there is no opponent whose choices respond to yours. It is a **single-player optimisation problem under chance**. In every situation, pick the action with the highest expected value. In 1956, Baldwin, Cantey, Maisel and McDermott worked out a nearly optimal playing strategy and published it in the *Journal of the American Statistical Association*.[14] Edward O. Thorp later refined it with computer calculations, and in *Beat the Dealer* (1962) he showed that keeping track of the cards already played can turn the edge in the player's favour.[15]

Card counting works because the deck has memory. Once cards are dealt, the cards left change the odds. In the **Hi-Lo** count (Harvey Dubner, 1963), cards 2–6 count +1, 7–9 count 0, and tens and aces count −1. The running total, divided by the number of decks left, is the **true count**. A high true count means the remaining cards are rich in tens and aces, which favours the player. The **Illustrious 18** are the 18 most valuable departures from basic strategy at given true counts (for example, take insurance at +3 or higher). They come from Don Schlesinger's *Blackjack Attack*.[16][17]

3 Finding the best play, and how the app does it

Basic strategy is the table of EV-maximising decisions for every player total against every dealer up-card, for a given set of rules. It is found by computing the expected value of each action exactly, working backwards over the cards still to come.[14] The **house edge** is the casino's expected profit, as a percentage of the initial bet, when the player follows basic strategy. Small rule changes move it measurably.[18]

IN THE APP

For the app's rules (6 decks, dealer **stands on soft 17** (S17), **double after split** allowed (DAS), **late surrender**, blackjack pays **3:2**), Card Solver reports a **house edge of 0.38%** under basic strategy. That means a player expectation of −0.38% of the initial bet per hand. This is in line with published house-edge calculators for these rules.[18]

Kuhn poker

IMPERFECT INFORMATION · SOLVED

1 The game, in plain words

Kuhn poker (Harold W. Kuhn, 1950) is a tiny poker game designed to be analysed exactly. The deck has three cards: King, Queen and Jack. Two players each put in 1 chip (the **ante**) and get one card; the third card is set aside unseen. Player 1 may **check** or **bet** 1. If player 1 checks, player 2 may check, which leads to a showdown, or bet. Whenever someone bets, the other player may **call** or **fold**. In a showdown, the higher card wins the pot.[19]

2 The game theory behind it

Kuhn poker is a two-player, zero-sum game of imperfect information. It is small enough to solve completely by hand. It has no pure-strategy equilibrium, meaning no equilibrium in which each player always takes the same action in the same spot. Equilibrium play requires **mixing**, that is, randomising between actions.[19]

−1/18	0 to 1/3	1/3
game value per hand to player 1 (player 2 wins +1/18)	player 1's equilibrium betting (bluffing) frequency with the Jack	player 2's equilibrium bluffing frequency with the Jack

In equilibrium, player 1 chooses a bluffing frequency α between 0 and 1/3 for betting with the Jack. Player 1 then bets the King with probability 3α, always checks the Queen, and calls with the Queen with probability α + 1/3 if player 2 bets after the check. Player 2 has a single equilibrium strategy. Always bet or call with the King. With the Queen, check if possible, and otherwise call one-third of the time. With the Jack, never call, and bet one-third of the time after a check.[19]

Why bluffing is rational. A player who bets only with the King gives their hand away, and the opponent can simply fold every time they bet. Bluffing with the worst card at the right rate means a bet no longer reveals the hand. The opponent's Queen then gains nothing from either calling or folding (it becomes **indifferent** between them), and this is exactly what stops the bluffer from being exploited. In game theory, bluffing is not a trick that relies on a bad opponent. It is part of optimal play.

3 Finding the best play, and how the app does it

With only 6 possible deals (3 cards for player 1 × 2 for player 2) and a handful of decisions, the equilibrium can be derived exactly. Write down the expected value of each strategy and solve for the frequencies that make the opponent indifferent. That small size is what makes Kuhn poker the classic worked example of a solved imperfect-information game.[19]

IN THE APP

Card Solver uses Kuhn poker to show equilibrium play exactly. The value −1/18 and the bluffing frequencies above are Kuhn's closed-form results, not estimates.

Video poker: Jacks or Better

1 PLAYER · EV OPTIMISATION

1 The game, in plain words

Video poker is a one-player machine game. You get five cards, choose which to **hold**, and the cards you discard are replaced from the rest of the 52-card deck. The final hand is paid from a fixed **pay table**. In "9/6" Jacks or Better, each coin bet pays 1 for a pair of jacks or better, 2 for two pair, 3 for three of a kind, 4 for a straight, 6 for a flush, 9 for a full house, 25 for four of a kind, 50 for a straight flush and 800 for a royal flush (when five coins are bet). The "9/6" refers to the full-house and flush payouts.[20][21]

2 The game theory behind it

There is no opponent, and the machine draws randomly from the remaining deck. Video poker is therefore a **single-player expected-value optimisation**. Each card can be held or discarded, so every deal offers 2^5 = **32** possible holds. The best play is simply the hold with the highest expected payout.

32	99.54%	47
possible holds for every deal	long-run return of 9/6 Jacks or Better with optimal play	unseen cards that the draws come from

3 Finding the best play, and how the app does it

For each of the 32 holds, average the payout over every possible set of replacement cards drawn from the 47 unseen cards. If you discard all five, there are 1,533,939 possible draws. Across all 32 holds there are 2,598,960 draw outcomes in total, which happens to equal the number of possible five-card hands. Then pick the hold with the highest average. With perfect play, 9/6 Jacks or Better returns **99.54%** of the money bet over the long run.[20]

IN THE APP

Card Solver evaluates all 32 holds by expected value and recommends the best one. For this game that is exact optimisation, not a heuristic.

Gin rummy

IMPERFECT INFORMATION · ZERO-SUM

1 The game, in plain words

Gin rummy is played by two players with ten cards each. On each turn, you draw a card, either the top card of the stock (face down) or the top card of the discard pile, and then discard one. You want to arrange your hand into **melds**: **sets** of three or four cards of the same rank, or **runs** of three or more consecutive cards in the same suit. Cards not in a meld are **deadwood**. They count as their pip value, with face cards worth 10 and aces worth 1. You may **knock** to end the hand when your deadwood is low enough, usually 10 or less, and going **gin** means having no deadwood at all. If the defender's deadwood is equal to or lower than the knocker's, that is an **undercut**, and the defender scores a bonus.[22]

2 The game theory behind it

Gin is a two-player, zero-sum game of **imperfect information** with chance. You can't see your opponent's hand or the stock. However, every card your opponent picks up from the discard pile, and every card they throw away, is evidence about what they are collecting. Good play mixes building your own melds, denying your opponent useful cards, and timing the knock. The full game is far too large to have a known equilibrium.

3 Finding the best play, and how the app does it

Two separate problems are involved. **Scoring a hand** is an exact optimisation. A card can be part of a set or a run, but not both, so the solver must search the possible ways to arrange the cards into melds and choose the one that leaves the least deadwood. **Choosing a play** (what to pick up, what to discard, when to knock) depends

on hidden information, and in practice relies on **heuristics**: rules of thumb that are usually good but not provably optimal.

IN THE APP

Card Solver's meld and deadwood calculations are **exact**. Its pick-up, discard and knock advice is a **heuristic**, not a game-theoretic solution.

Cribbage

IMPERFECT INFORMATION · ZERO-SUM

1 The game, in plain words

In two-player cribbage, each player is dealt six cards and puts two face down into the **crib**, an extra hand that belongs to the dealer. A **starter** card is then cut from the deck. Players take turns laying cards in the **play** (pegging), scoring as they go, without the running total going over 31. Afterwards, each hand is counted together with the starter. Points come from combinations that add up to 15 (2 points each), pairs, runs, flushes, and "his nobs" (the jack of the same suit as the starter, worth 1). The first to reach the target, usually 121, wins.[23]

2 The game theory behind it

Cribbage is a two-player, zero-sum game with chance and imperfect information. The discard is its signature decision. With six cards you have C(6,2) = **15** ways to choose the two cards to throw, and the starter will be one of the **46** cards you have not seen. The cards you keep score for you. The cards you throw score for the dealer, which is you on your own deal and your opponent otherwise. A good discard maximises **expected value**: the average points of your kept hand, plus the crib if it is yours, or minus the crib if it is your opponent's.

3 Finding the best play, and how the app does it

For each of the 15 discards, average the score of the four kept cards with each possible starter. Then estimate the crib's value, which also depends on the two cards your opponent throws in. Those cards are unknown, so a model is needed: assuming random cards is the simplest neutral choice. A real opponent throws strategically, so in reality the crib value will differ from this estimate.

IN THE APP

Card Solver ranks discards by expected value, averaging over the starter card and the crib. It **assumes your opponent throws random cards** into the crib. This is a simplifying assumption, not a model of a strategic opponent.

At a glance

Game	Players / information	Chance	Theoretical status	What the app computes
Poker (hold'em)	2+ players, imperfect information, zero-sum	Yes	Heads-up limit essentially weakly solved (2015); others unsolved	Simplified, labelled fold/call/raise model; not GTO
Blackjack	Player vs fixed dealer rules	Yes	Optimisation problem; basic strategy known since 1956	Basic strategy; 0.38% house edge for its rules
Kuhn poker	2 players, imperfect information, zero-sum	Yes	Solved exactly; value −1/18 to player 1	Exact equilibrium
Video poker (9/6 JoB)	1 player vs pay table	Yes	Optimisation; 99.54% return with optimal play	Exact EV over all 32 holds
Gin rummy	2 players, imperfect information, zero-sum	Yes	Unsolved	Exact melds and deadwood; heuristic advice
Cribbage	2 players, imperfect information, zero-sum	Yes	Unsolved	EV discard over starter and crib; random-opponent crib assumption

DETAILS-PAGE BLURB

Card Solver covers six card games, from pure expected-value problems like video poker and blackjack to games of hidden information like poker, gin rummy and cribbage. It finds the exact best hold in Jacks or Better and applies basic strategy in blackjack. In Kuhn poker it shows the exact equilibrium that makes bluffing mathematically correct. Where a game is too large to solve, the app says so and labels its advice as a simplified model or heuristic.

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