PUZZLE SUITE · GAME THEORY NOTES

The Game Theory Behind Connect Solver

Solved games of perfect information: Connect 4 and tic-tac-toe

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A short game-theory primer

Game theory is the mathematics of decisions whose outcome depends on choices, chance, or other people. The ideas below come up throughout this page.

Perfect vs imperfect information

In a game of **perfect information**, every player sees everything that has happened so far, as in chess or Connect 4. In a game of **imperfect information**, something is hidden, such as an opponent's cards in poker. [1]

Zero-sum

A game is **zero-sum** when one player's gain is exactly the other's loss. Most two-player board games and heads-up poker are zero-sum. [2]

Minimax

Minimax means choosing the move whose *worst-case* result is best, on the assumption that the opponent always replies as well as possible. Von Neumann's 1928 minimax theorem showed that every finite two-player zero-sum game has a well-defined value when players are allowed to randomise. [3][4]

Nash equilibrium

A **Nash equilibrium** is a set of strategies in which no player can do better by changing only their own strategy. John Nash proved in 1950 that every finite game has at least one, possibly in mixed (randomised) strategies. In poker this is what "GTO" (game-theory optimal) refers to. [5][6]

Expected value (EV)

The **expected value** is the probability-weighted average outcome of a decision: what it would earn per play over many repetitions. When there is no opponent to outwit, the best play is usually just the play with the highest EV. [7]

Solved games

Following Allis, a game is **ultra-weakly solved** when its result under perfect play is known, **weakly solved** when a strategy that achieves that result from the start is known, and **strongly solved** when perfect play is known from every legal position. [8][9]

Connect 4

2 PLAYERS · ZERO-SUM · SOLVED

1 The game, in plain words

Connect 4 is played on an upright board with 7 columns and 6 rows. Two players take turns dropping a disc of their colour into a column, and it falls to the lowest empty space. The first to line up **four in a row**, horizontally, vertically or diagonally, wins. If all 42 spaces fill up with no line of four, the game is a draw. [10]

2 The game theory behind it

Connect 4 is a two-player, zero-sum game of perfect information with no chance, so by Zermelo's theorem it has a definite value.[11] It was solved in 1988, independently, by James D. Allen (1 October) and Victor Allis (16 October). With perfect play, **the first player wins**, but only by starting in the **centre column**.[10][12] John Tromp later built a database giving the result of every position with 8 discs, and the game is now strongly solved.[13][9]

4,531,985,219,092
legal positions on the 7×6 board

1988
solved by Allen and by Allis, independently

Centre
the only winning first move

First move (column)	1	2	3	4 (centre)	5	6	7
Result for the first player, with perfect play	Lose	Lose	Draw	**Win**	Draw	Lose	Lose

3 Finding the best play, and how the app does it

Unlike chess, Connect 4 is small enough to search all the way to the end, so a solver can give exact results instead of estimates. The standard techniques, set out in Pascal Pons's tutorial, are:[14]

- **Negamax**: a compact form of minimax for zero-sum games. One player's score is the negative of the other's, so a single "maximise my score" routine serves both sides.[15]
- **Alpha-beta pruning**: skip branches that cannot change the result. Trying central columns first makes the pruning much more effective.[16]
- **Transposition table**: remember positions already solved, since different move orders often reach the same position.[17]
- **Opening book**: precomputed results for early positions, where the remaining search is largest.

IN THE APP

Connect Solver has a **perfect solver** based on Pascal Pons's techniques: negamax with alpha-beta pruning, a transposition table, and an **opening book** covering all positions with up to 8 pieces. Its results for the seven possible first moves are lose, lose, draw, win, draw, lose, lose, which match the known solution.[14][18]

Tic-tac-toe

2 PLAYERS · ZERO-SUM · STRONGLY SOLVED

1 The game, in plain words

Two players, X and O, take turns marking empty cells of a 3×3 grid, with X going first. The first player to get three marks in a row, column or diagonal wins. If all nine cells fill up with no line, the game is a draw.[19]

2 The game theory behind it

Tic-tac-toe is the smallest well-known example of a two-player, zero-sum game of perfect information. It is **strongly solved**: with perfect play by both sides, it is always a **draw**.[19] There are **255,168** possible games,

counting every distinct move sequence that ends in a win or a full board, a figure that can be checked by exhaustively enumerating the game tree. That falls to 26,830 once rotations and reflections are treated as the same, and there are only 765 essentially different positions.[19]

255,168	765	Draw
possible games (distinct move sequences)	essentially different positions	result with perfect play

3 Finding the best play, and how the app does it

Because the whole game tree is so small, plain **minimax** can search every line to the end almost instantly, with no evaluation estimates needed. This makes tic-tac-toe the classic first example of the idea Connect 4 builds on: score every final position, then work back up the tree assuming both players choose their best reply.[4]

IN THE APP

Tic-tac-toe is small enough to search completely, so the app's analysis matches the known result: a draw with perfect play.

At a glance

Game	Type	Size	Solved	Value	App approach
Connect 4 (7×6)	2 players, zero-sum, perfect information, no chance	`4,531,985,219,092 positions`	1988 (Allen; Allis); strongly solved since	First-player win (centre opening)	Negamax, alpha-beta, transposition table, 8-piece opening book
Tic-tac-toe	2 players, zero-sum, perfect information, no chance	`255,168 games`	Strongly	Draw	Complete search

DETAILS-PAGE BLURB

Connect Solver plays Connect 4 perfectly. It combines negamax search, alpha-beta pruning, a transposition table and an opening book of every position up to 8 pieces, based on Pascal Pons's techniques. It confirms the 1988 result that the first player wins only by opening in the centre column. The app also covers tic-tac-toe, the textbook solved game, which is always a draw with perfect play.

References

1. Wikipedia. "Perfect information." https://en.wikipedia.org/wiki/Perfect_information
2. Wikipedia. "Zero-sum game." https://en.wikipedia.org/wiki/Zero-sum_game
3. von Neumann, J. (1928). "Zur Theorie der Gesellschaftsspiele." *Mathematische Annalen* 100, 295–320. https://doi.org/10.1007/BF01448847

4. Wikipedia. "Minimax." https://en.wikipedia.org/wiki/Minimax
5. Nash, J. F. (1950). "Equilibrium points in n-person games." *PNAS* 36(1), 48–49. https://doi.org/10.1073/pnas.36.1.48
6. Wikipedia. "Nash equilibrium." https://en.wikipedia.org/wiki/Nash_equilibrium
7. Wikipedia. "Expected value." https://en.wikipedia.org/wiki/Expected_value
8. Allis, L. V. (1994). *Searching for Solutions in Games and Artificial Intelligence*. PhD thesis, University of Limburg (Maastricht). https://project.dke.maastrichtuniversity.nl/games/files/phd/SearchingForSolutions.pdf
9. Wikipedia. "Solved game." https://en.wikipedia.org/wiki/Solved_game
10. Wikipedia. "Connect Four" (solution history; 4,531,985,219,092 positions). https://en.wikipedia.org/wiki/Connect_Four
11. Wikipedia. "Zermelo's theorem (game theory)." https://en.wikipedia.org/wiki/Zermelo%27s_theorem_(game_theory)
12. Allis, L. V. (1988). *A Knowledge-based Approach of Connect-Four*. M.Sc. thesis, Vrije Universiteit Amsterdam. (Summary via Chess Programming Wiki.) https://www.chessprogramming.org/Victor_Allis
13. Tromp, J. "John's Connect Four Playground" (8-ply opening database, 1995). https://tromp.github.io/c4/c4.html
14. Pons, P. "Solving Connect 4: how to build a perfect AI." gamesolver.org. http://blog.gamesolver.org/solving-connect-four/01-introduction/
15. Wikipedia. "Negamax." https://en.wikipedia.org/wiki/Negamax
16. Knuth, D. E. & Moore, R. W. (1975). "An analysis of alpha-beta pruning." *Artificial Intelligence* 6(4), 293–326. https://doi.org/10.1016/0004-3702(75)90019-3
17. Chess Programming Wiki. "Transposition Table." https://www.chessprogramming.org/Transposition_Table
18. Pons, P. Connect 4 solver source code. GitHub. https://github.com/PascalPons/connect4
19. Wikipedia. "Tic-tac-toe." https://en.wikipedia.org/wiki/Tic-tac-toe