PUZZLE SUITE · GAME THEORY NOTES

The Game Theory Behind Cube Solver

Single-player puzzles, group theory, and the search for the shortest solve

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A short game-theory primer

Game theory is the mathematics of decisions whose outcome depends on choices, chance, or other people. The ideas below come up throughout this page.

Perfect vs imperfect information

In a game of **perfect information**, every player sees everything that has happened so far, as in chess or Connect 4. In a game of **imperfect information**, something is hidden, such as an opponent's cards in poker. [1]

Zero-sum

A game is **zero-sum** when one player's gain is exactly the other's loss. Most two-player board games and heads-up poker are zero-sum. [2]

Minimax

Minimax means choosing the move whose *worst-case* result is best, on the assumption that the opponent always replies as well as possible. Von Neumann's 1928 minimax theorem showed that every finite two-player zero-sum game has a well-defined value when players are allowed to randomise. [3][4]

Nash equilibrium

A **Nash equilibrium** is a set of strategies in which no player can do better by changing only their own strategy. John Nash proved in 1950 that every finite game has at least one, possibly in mixed (randomised) strategies. In poker this is what "GTO" (game-theory optimal) refers to. [5][6]

Expected value (EV)

The **expected value** is the probability-weighted average outcome of a decision: what it would earn per play over many repetitions. When there is no opponent to outwit, the best play is usually just the play with the highest EV. [7]

Solved games

Following Allis, a game is **ultra-weakly solved** when its result under perfect play is known, **weakly solved** when a strategy that achieves that result from the start is known, and **strongly solved** when perfect play is known from every legal position. [8][9]

The Cube, 2×2 to 7×7

SINGLE-PLAYER · PERFECT INFORMATION

1 The game, in plain words

The cube is a puzzle made of smaller pieces ("cubies") whose outer faces carry coloured stickers. Any face layer, and on larger cubes any inner layer, can be turned. A solved cube has one colour per face. You scramble it with random turns, and the goal is to bring every face back to a single colour. The Cube Solver handles sizes from 2×2 up to 7×7. [10]

A **move** needs a precise definition, because "how many moves" depends on it. In the **half-turn metric** (HTM), turning one face by a quarter or a half turn counts as one move. In the **quarter-turn metric** (QTM), only quarter turns count as one, so a half turn costs two.

2 The game theory behind it

A cube has **one player and no opponent**, and nothing is hidden or left to chance once the scramble is in front of you. In game-theory terms it is a single-player puzzle of perfect information. Since nobody is pushing back, minimax and Nash equilibrium do not apply. What matters is the structure of the puzzle itself.

That structure comes from **group theory**, the algebra of symmetries. Every sequence of turns rearranges the stickers, and these rearrangements form a mathematical group: any two can be combined, every one can be undone, and doing nothing is also allowed. Each reachable position matches exactly one element of the group, so "how many positions are there?" becomes "how big is the group?"[11]

3,674,160
positions of the 2×2 ($7! \times 3^6$)

$\approx 4.3 \times 10^{19}$
positions of the 3×3: exactly 43,252,003,274,489,856,000

11 & 20
God's number (HTM) for the 2×2 and the 3×3

The 3×3 count comes from arranging 8 corners and 12 edges, with two restrictions. The orientation of the last corner and of the last edge is forced by the others, and corner and edge permutations must have matching parity. That gives $8! \times 3^7 \times 12! \times 2^{11} \div 2$.[11] **God's number** is the largest number of moves an optimal solution can ever need, in other words the worst case over all positions. For the 2×2 it is 11 in the half-turn metric (14 in quarter turns).[12] For the 3×3, Rokicki, Kociemba, Davidson and Dethridge proved in 2010 that it is 20 in the half-turn metric. They did it by splitting all positions into cosets (large structured subsets) and solving every one by computer.[13][14]

3 Finding the best play, and how the app does it

Solving a cube means searching for a path from the scrambled position back to the solved one. Because the number of positions is huge, the methods differ in how they trade speed against getting the shortest possible solution.

Optimal search with IDA*. IDA* (iterative-deepening A*, Korf 1985) is a depth-first search that repeats with an ever larger move budget. It uses a **heuristic**, an estimate of the moves still needed. If the heuristic is **admissible**, meaning it never overestimates, the first solution IDA* finds is guaranteed to be as short as possible.[15] Korf applied this to the 3×3 in 1997, using precomputed **pattern databases** (tables of exact distances for parts of the cube) as heuristics.[16]

Kociemba's two-phase algorithm. Phase 1 searches for moves that bring the cube into a special subgroup, usually written G1. In G1 every piece is correctly oriented, and the four middle-layer edges sit in the middle layer. From there, the cube can be solved using only the moves U, D, R2, L2, F2 and B2. Phase 2 then finishes the solve inside G1. Each phase searches a far smaller space than the full cube, so the method returns short solutions very quickly. They are not guaranteed to be the shortest possible.[17]

Reduction for big cubes. On 4×4 and larger cubes, the standard approach is to first build solid centre blocks, then pair up matching edge pieces. After that, the cube can be turned like a 3×3 and solved as one. Even-sized

cubes can also show **parity** cases, positions that cannot occur on a real 3×3, and these need extra algorithms. [18][19]

IN THE APP

- **2×2:** optimal IDA* search, so it returns a shortest solution.
- **3×3:** Kociemba's two-phase algorithm, which gives fast, short solutions that are not guaranteed to be the shortest.
- **4×4 to 7×7:** reduction to a 3×3-like state, then a 3×3 solve. These solutions are practical, not move-optimal.

At a glance

Puzzle	Players / information	Positions	God's number (HTM)	App method	Optimal?
2×2	1 player, perfect information, no chance	3,674,160	11 [12]	IDA* search	Yes
3×3	1 player, perfect information, no chance	$\approx 4.3 \times 10^{19}$	20 (proven 2010) [13]	Kociemba two-phase	Near-optimal, not guaranteed
4×4 – 7×7	1 player, perfect information, no chance	Far larger (not quoted here)	Not known	Reduction, then 3×3 solve	No (practical)

DETAILS-PAGE BLURB

Cube Solver solves cubes from 2×2 to 7×7. It is built on the group theory that makes the cube a single-player puzzle with exactly 43,252,003,274,489,856,000 positions at the 3×3 size. The 2×2 is solved optimally with IDA* search, and the 3×3 with Kociemba's fast two-phase algorithm. Larger cubes are reduced to a 3×3 and finished the same way.

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