PUZZLE SUITE · GAME THEORY NOTES

The Game Theory Behind Sudoku Solver

Logic, search and constraint satisfaction, including Killer, Diagonal and mini variants

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A short game-theory primer

Game theory is the mathematics of decisions whose outcome depends on choices, chance, or other people. The ideas below come up throughout this page.

Perfect vs imperfect information

In a game of **perfect information**, every player sees everything that has happened so far, as in chess or Connect 4. In a game of **imperfect information**, something is hidden, such as an opponent's cards in poker.[1]

Zero-sum

A game is **zero-sum** when one player's gain is exactly the other's loss. Most two-player board games and heads-up poker are zero-sum.[2]

Minimax

Minimax means choosing the move whose *worst-case* result is best, on the assumption that the opponent always replies as well as possible. Von Neumann's 1928 minimax theorem showed that every finite two-player zero-sum game has a well-defined value when players are allowed to randomise.[3][4]

Nash equilibrium

A **Nash equilibrium** is a set of strategies in which no player can do better by changing only their own strategy. John Nash proved in 1950 that every finite game has at least one, possibly in mixed (randomised) strategies. In poker this is what "GTO" (game-theory optimal) refers to.[5][6]

Expected value (EV)

The **expected value** is the probability-weighted average outcome of a decision: what it would earn per play over many repetitions. When there is no opponent to outwit, the best play is usually just the play with the highest EV.[7]

Solved games

Following Allis, a game is **ultra-weakly solved** when its result under perfect play is known, **weakly solved** when a strategy that achieves that result from the start is known, and **strongly solved** when perfect play is known from every legal position.[8][9]

Classic Sudoku

1 PLAYER · CONSTRAINT SATISFACTION

1 The game, in plain words

A sudoku is a 9×9 grid divided into nine 3×3 boxes. Some cells start filled in; these are the **clues** (or "givens"). You fill in every empty cell with a digit from 1 to 9 so that each row, each column and each 3×3 box contains every digit exactly once. A well-made puzzle has exactly one solution.[10]

2 The game theory behind it

Sudoku has no opponent and no chance, so it is **not an adversarial game**. Minimax and Nash equilibrium have nothing to work on. The useful theory here is about logic and search. Sudoku is a **constraint satisfaction problem** (CSP): a set of variables (the 81 cells), each with a set of possible values (the digits 1 to 9), and a set of rules that limit which combinations are allowed (no repeated digit in any row, column or box).[11]

17	6.67×10^{21}	NP-complete
minimum number of clues for a uniquely solvable 9×9 sudoku	completed 9×9 grids (6,670,903,752,021,072,936,960)	status of generalised $n^2 \times n^2$ sudoku

How hard is sudoku, in general? The question "does this puzzle have a solution?", asked for sudoku grids of any size ($n^2 \times n^2$ cells), is **NP-complete**. In plain terms, it belongs to a large family of problems for which no known algorithm scales efficiently to arbitrarily large instances, although any proposed solution is quick to check. The result is due to Yato and Seta (2003).[12] This does not make an ordinary 9×9 puzzle hard for a computer. The standard grid is small, and solvers finish it almost instantly.

How few clues can there be? In 2012, Gary McGuire, Bastian Tugemann and Gilles Civario announced an exhaustive computer search proving that no 16-clue sudoku has a unique solution. Since 17-clue puzzles were already known, **17 is the minimum**.[13][14]

3 Finding the best play, and how the app does it

Two ideas, used together, solve almost any sudoku quickly:

- **Constraint propagation.** Track the candidate digits for every cell and keep applying the rules. If a cell has only one candidate left, fill it in. If a digit fits in only one place in a row, column or box, it goes there. Every placement removes candidates from neighbouring cells, which often sets off a chain of further placements.[15]
- **Backtracking search.** When the logic stalls, pick a cell (ideally one with the fewest candidates), try a value, and continue. If that leads to a contradiction, undo it and try the next value. This is a depth-first search through partial grids, and combined with propagation it rarely needs to guess more than a few times.[15][11]

Other exact methods exist as well. For example, Knuth's "Dancing Links" technique treats sudoku as an exact-cover problem.[16]

IN THE APP

Sudoku Solver treats each puzzle as a constraint satisfaction problem. It solves with constraint propagation and backtracking, the same approach as the Classic grid, adapted to each variant's extra rules.

Variants: Killer, Diagonal, Minis

1 PLAYER · ADDED CONSTRAINTS

1 The game, in plain words

- **Killer sudoku:** the grid is overlaid with outlined groups of cells called **cages**, each labelled with a sum. The digits in a cage must add up to that sum, and no digit may repeat within a cage. The usual row, column and

box rules still apply. Killer puzzles often start with few or no given digits.[17]

- **Diagonal sudoku** (also called Sudoku X): both main diagonals must also contain each digit from 1 to 9 exactly once.[10]
- **Mini sudoku:** smaller grids that use the same idea. Every row, column and box contains each digit once, with fewer digits and smaller boxes, for example 4×4 grids with 2×2 boxes or 6×6 grids with 3×2 boxes.[10]

2 The game theory behind it

The variants change the **constraints**, not the nature of the problem. They are still single-player constraint satisfaction puzzles with no opponent and no chance. Diagonal sudoku adds two more "all different" groups. Killer sudoku adds **sum constraints**, which bring in arithmetic reasoning. For example, a two-cell cage that sums to 3 can only contain 1 and 2. Minis have fewer cells and digits, so their search spaces are tiny.

3 Finding the best play, and how the app does it

The same propagate-then-search approach applies; only the rule set changes. For a Killer cage, the solver can list the digit combinations that reach the cage's sum without repeating a digit, and rule out any candidate that appears in none of them. For a diagonal, each diagonal is checked like an extra row. Minis use smaller digit sets and box shapes.[15][17]

IN THE APP

Killer, Diagonal and mini puzzles share the same solving engine as the Classic grid. Each variant adds its own constraints (cage sums, diagonals, smaller boxes) to the rule set that drives propagation and backtracking.

At a glance

Puzzle	Type	Extra constraints	App approach
Classic 9×9	1 player, logic / constraint satisfaction, no chance	Rows, columns, 3×3 boxes	Constraint propagation + backtracking
Killer	Same	+ cage sums, no repeats within a cage	Same engine + sum constraints
Diagonal	Same	+ both main diagonals	Same engine + two extra groups
Minis	Same	Smaller grids and boxes	Same engine, smaller digit set

DETAILS-PAGE BLURB

Sudoku Solver treats every puzzle as a constraint satisfaction problem, a logic search rather than a contest against an opponent. It combines constraint propagation with backtracking to solve classic, Killer, Diagonal and mini grids. In its general form sudoku is NP-complete, and a 9×9 puzzle with a unique solution needs at least 17 clues, a result proved by exhaustive computer search in 2012.

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